GROUPS WITH DENSE NORMAL SUBGROUPS

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ABSTRACT

A group is said to have dense normal subgroups, if each non-empty open interval in its lattice of subgroups contains a normal subgroup. The structure of this and related classes of groups is investigated. Typical results are: an infinite group with dense ascendant subgroups is locally nilpotent: a non-torsion group with dense normal subgroups is abelian, etc.

There are a large number of results dealing with the structure of groups with many normal subgroups. The word "many" is, of course, given to many interpretations. In this paper we interpret it to mean, roughly, that the lattice of normal subgroups is dense in the lattice of all subgroups, under the interval topology. Accordingly, we give the following ($\subset$ denotes always proper containment).

Definition. A group G is said to have *dense normal subgroups* if, whenever $H \subset K \subseteq G$, H and K subgroups of G and H is not maximal in K, there exists a normal subgroup N of G such that $H \subset N \subseteq K$.

Obviously, one can modify this definition by changing the word "normal" to some other group property, and thus obtain other classes of groups.

In Section 1, we determine completely the finite groups with dense subnormal subgroups. There are all meta-nilpotent, and generally even nilpotent. In Section 2 we show that an infinite group with dense ascendant subgroups is locally nilpotent. In Section 3 we deal with infinite groups with dense normal subgroups. If such a group contains an element of infinite order, it is abelian, while an infinite torsion group of this type is either a Dedekind group or an extension of a central subgroup of type $C(p^\infty)$ by a finite Dedekind group. In Section 4 we prove that an infinite group with dense quasi-normal subgroups is locally nilpotent. Moreover, if such a group contains an element of infinite order, it is quasi-Hamiltonian. In dealing with quasi-normal subgroups, we prove a result (Theorem 6) of independent interest. It states that a group generated by quasi-normal subgroups, which are finite nilpotent or infinite cyclic, is locally nilpotent.

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Notation and terminology. If G is a group, a subgroup H is called *ascendant*, if H can be connected to G by means of an increasing well-ordered normal series. If this series is finite, H is called subnormal, denoted $H \lhd\lhd G$, while $H \lhd G$ denotes that H is normal in G. H is *quasi-normal*, if it permutes with all other subgroups. It is known that maximal quasi-normal subgroups are normal [7, Theorem 19, p. 438], therefore quasi-normal subgroups of finite groups are subnormal. H is *n-maximal* in G, if there exists a chain $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$, with each H_i a maximal subgroup of H_{i+1}. A group G is a *Dedekind* group, if all its subgroups are normal, and *quasi-Hamiltonian*, if all its subgroups are quasi-normal. For the structure of these two classes of groups see, respectively, [3, Theorem 12.5.4] and [9, Chapter 1, Sections 4, 5]. A group is said to satisfy the *normalizer condition*, if each subgroup is different from its normalizer. This is equivalent to each subgroup being ascendant. A *section* of a group G is a group H/K, where $K \subseteq H \subseteq G$ and $K \lhd H$.

$Z(G)$ denotes the center of the group G. $R(G)$ denotes the Hirsch-Plotkin radical of G, which is the unique maximal normal locally nilpotent subgroup of G. For finite groups, $R(G) = F(G)$, the Fitting subgroup of G. If A is a subset of G, $\langle A \rangle$ is the subgroup generated by A.

1. THEOREM 1. *Let G be a finite group with dense subnormal subgroups. Then one of the following three possibilities holds:*

a. *G is nilpotent.*

b. *There exists two different primes, p and q, such that $G = PQ$, where P is a p-group, which is a minimal normal subgroup of G. Q is a q-group, of one of the following types:*

1. *Cyclic.*
2. *A direct product of a cyclic group and a group of order q.*
3. *The group $\langle a, b \mid a^{q^{n-1}} = b^q = k,\ b^{-1}ab = a^{1+q^{n-2}} \rangle$.*
4. *The quaternion group.*

The centralizer of P in Q is a maximal subgroup of Q. In cases 2 and 3, it is the unique non-cyclic maximal subgroup of Q.

c. *$G = H \times C_s$, where H is of type b. with cyclic Q, C_s is cyclic of prime order s and $s \neq q$.*

Conversely, a group of each of these three types has dense subnormal subgroups.

Proof. First, G is solvable. Indeed, if G is not of prime order, by assumption G contains some proper normal subgroup, N say. By induction both N and G/N are solvable hence so is G.

Assume that G is not nilpotent. Then G contains some maximal subgroup,

M say, which is not normal. Let K be maximal in M. Then K is contained in some proper normal subgroup of G, L say and $K = L \cap M$. Hence $K \lhd M$. Each maximal subgroup of M being normal, M is nilpotent. Since M is self-normalizing, M is a Carter subgroup of G, [1].

Let $|G:M| = p^n$, p a prime, and let G_p be a Sylow p-subgroup of G. Let L be a maximal subgroup of G containing G_p. If L is not normal, then, as for M, we see that L is a Carter subgroup, and therefore conjugate to M[1] which is impossible. Hence $L \lhd G$. This implies $G_p \lhd G$ (otherwise, taking $L \supseteq N(G_p)$ yields a contradiction). Also $G = MG_p$.

Let H be 2-maximal in M. By assumption, $H \subset K \subset M$, where $K \lhd\lhd G$ and K is maximal in M. If K is the unique maximal subgroup of M, M is a cyclic q-group, for some prime q. If M has more than one maximal subgroup, let L be maximal in M and $L \neq K$. If $L \lhd\lhd G$, then $M \lhd\lhd G$, impossible. It follows that all 2-maximal subgroups of M are contained in K. Since $L \nsubseteq K$, L is not generated by 2-maximal subgroups of M, i.e. L is not generated by its maximal subgroups. This is possible only if L has a unique maximal subgroup, and hence L is a cyclic q-group, q a prime.

Let $|M:L| = s$ (a prime). If $s \neq q$, then $M = L \times M_s$, M_s being a Sylow s-subgroup of M. If $s = q$, then, working through the list of q-groups with cyclic maximal subgroup, one shows that those groups mentioned under b are the only ones with the property, that they contain a maximal subgroup containing all 2-maximal subgroups. Moreover, in cases b.2 and b.3 this maximal subgroup is the unique noncyclic one.

Let us denote $Q = M$ if $s = q$, or if M is a cyclic q-group and $Q = L$ otherwise. Now L is also contained in some maximal normal subgroup of G, N say. Since $M \neq N$, $G = MN$. Let $P = G_p \cap N$, then P is the Sylow p-subgroup of N, and $P \lhd G$. By order considerations (recalling $|G:M| = p^n$), $G = MP$.

Assume $s = p$ (hence $s \neq q$, as G is not a p-group). Let $Z = Z(G_p)$. Then $Z \lhd G$, and M normalizes ZM_s. By maximality of M, either $ZM_s = M_s$, which implies $Z = M_s$ (since $|M_s| = s = p$) or $MZM_s = MZ = G$ $(= MG_p)$. The last equation implies $|G_p:Z| \leqq p$, hence G_p/Z is cyclic, and $G_p = Z$ is abelian. In any case, M_s centralizes G_p. As M is nilpotent, M_s centralizes also M. Hence $M_s \lhd G$.

Assume $s \neq p$ and $s \neq q$. Then M_s is normal in K, hence subnormal in G. M_s being now a Sylow subgroup of G, we again obtain $M_s \lhd G$.

If $p = q$, G is either a p-group, or $G = M_s \times G_p$, and is nilpotent in either case. Hence $p \neq q$. Therefore $P \cap Q = 1$, and $P \cap M \subseteq M_s$. If $P \cap M \neq 1$, then $P \cap M = M_s$, hence $N \supseteq LM_s = M$, a contradiction. Thus $P \cap M = 1$. Therefore either $G = PQ$ (when $M = Q$) or $G = M_s \times PQ$. By maximality of M, P is a minimal normal subgroup. M does not centralize P, since G is not nilpotent. However, K does centralize P, being equal to either $O_q(G)$ or $M_s \times O_q(G)$ This ends the proof of Theorem 1.

We omit the verification of the converse part.

COROLLARY. *Let G be a finite group with dense subnormal subgroups. If H is an n-maximal subgroup of G, for some $n \geqq 3$, then H is nilpotent and subnormal in G.*

Proof. We need consider only cases b. and c. in Theorem 1. We shall give the details only for case b., the other one being very similar to it. Let, then, $G = PQ$, with P and Q as in b. Let $E = C_Q(P)$, then it is obvious that $F(G) = E \times P$, and we are given that $|Q:E| = q$. Let H be any subgroup of G. If $H \nsubseteq F(G)$, then H is nilpotent, with $F(G)$. H is also subnormal in the nilpotent group $F(G)$, and since $F(G) \lhd G$, $H \lhd\lhd G$. We may assume, then, that $H \nsubseteq F(G)$. Write $H = H_pH_q$, where H_p and H_q are the p-Sylow and a q-Sylow subgroups of H, respectively. Then $H_p \subseteq P$, and we may assume that $H_q \subseteq Q$. Since $H \nsubseteq F(G)$, $H_q \nsubseteq E$. As E contains all subgroups of Q of index q^2, $H_q = Q$ or H_q is a maximal subgroup of Q different from E. In any case, $Q = H_qE$, so H_q induces on P the same automorphism group as Q. Since Q is irreducible on P, it follows that the p-Sylow subgroup in any subgroup containing H_q is 1 or P. Hence the only subgroups containing H_q are H_q, H_qP, Q and G. Hence H_q, and also H, cannot be n-maximal, for any $n \geqq 3$.

2. THEOREM 2. *An infinite group with dense ascendant subgroups is locally nilpotent. Moreover, each finitely generated subgroup of such a group is ascendant.*

Proof. Let G be a group with dense ascendant subgroups. Suppose we know already that G is locally nilpotent, and let H be any finitely generated subgroup of G. One can find another finitely generated subgroup, K, such that $H \subset K$ and H is not maximal in K. By assumption, there exists an ascendant subgroup of G, L, such that $H \subset L \subset K$. As G is locally nilpotent, L is nilpotent, therefore H is subnormal in L and ascendant in G. Hence the second statement of the theorem follows from the first. To prove that G is locally nilpotent, we assume first that G is a torsion group. We shall begin by proving that G contains a cyclic ascendant subgroup.

Let H be a finite subgroup of G. If $p^2 \mid |H|$, for some prime p, then H contains a subgroup P of order p^2. By assumption, an ascendant subgroup of G lies between $\langle 1 \rangle$ and K, and this subgroup is cyclic (of order p). On the other hand, if $|H|$ is square-free, but not prime, then by [3, Corollary 9.4.1] H contains some subgroups of order pq, p and q primes, and therefore G contains some ascendant subgroup of order either p or q.

We may assume, then, that each finite subgroup of G has prime order.

Let $1 \neq a \in G$, let p be the order of a, let $b \in G$, $b \notin \langle a \rangle$, let q be the order of b, and denote $H = \langle a, b \rangle$. Then H is infinite.

We shall construct, by induction, a properly decreasing sequence $\{H_i\}$ of subgroups of H, each of them normal and of finite index in H.

First, define $H_0 = H$. Next, suppose H_i has already been found. As a subgroup of finite index in the finitely generated group H, H_i is also finitely generated. Therefore H_i contains some maximal normal subgroup, K say.

Assume K is not maximal in H_i. Then some subgroup L, lying properly between H_i and K, is ascendant in G, hence also in H_i. Let $\{L_\alpha\}$ be an ascending normal series from L to H_i, and let $L_\beta = H_i$ be its last term. Since H_i is finitely generated, β is not a limit ordinal. Hence $L_{\beta-1}$ is a normal subgroup of H_i properly containing K, a contradiction. Thus K is a maximal subgroup of H_i.

Being maximal and normal, the index of K in H_i is prime. Therefore $|H:K|$ is finite, and thus K contains some subgroup which is normal and of finite index in H_i. This subgroup can be taken as H_{i+1}.

The groups H/H_i are all finite groups with dense subnormal subgroups. Moreover, each of these groups is, with H, generated by two elements, g_i and h_i say, of order p and q respectively, and each element of H/H_i has a prime order.

Each group H/H_i is of the type described in Theorem 1. Suppose first it is of type b. Then $H/H_i = PQ$, as in Theorem 1. Q is a homomorphic image of PQ, so, being an r-group for some prime r, Q has exponent r, which shows, using Theorem 1, that $|Q| \leqq r^2$. Here $r = p$ or $r = q$. P is acted irreducibly upon by a group of order r, hence its order must be s^n, for some prime s, with n being the order of s (mod. r). If H/H_j, for $j > i$, is also of type b., then r and s must be same for H/H_j and H/H_i, because $|H:H_i| \,|\, |H:H_j|$. Hence $|H:H_j| \leqq r^2 s^n$, so there are only a finite number of groups of type b. of Theorem 1 among the groups H/H_j. In almost the same way, one proves that only a finite number of the H/H_i's are of type c. Suppose H/H_i is nilpotent. If $p \neq q$, then H/H_i is necessarily the abelian group of order pq. If $p = q$, H/H_i is a finite group, with two generators and of exponent p. There are only a finite number of such groups by a fundamental result of Kostrikin [4]. Hence there are only finitely many of the H/H_i's, an obvious contradiction.

This proves that G contains some ascendant cyclic subgroup. This, in turn, implies that the Hirsch-Plotkin radical, $R(G)$, of G, is non-trivial (e.g. [2, Theorem 2]).

Define a sequence $\{R_\alpha\}$ of subgroups of G as follows:

$$R_0 = \langle 1 \rangle, \; R_{\alpha+1}/R_\alpha = R(G/R_\alpha), \; R_\lambda = \bigcup_{\alpha<\lambda} R_\alpha$$

if λ is a limit ordinal. Since each factor group G/R_α satisfies the same conditions as G, $R_{\alpha+1} \neq R_\alpha$ unless $G = R_\alpha$. Therefore $G = R_\mu$ for some ordinal μ. As each group $R_{\alpha+1}/R_\alpha$ is locally nilpotent and torsion, therefore locally finite, G itself is locally finite.

Let H be any finite subgroup of G. One can find another finite subgroup, K, such that $H \subset K$, and H is n-maximal in K, where $n \geqq 3$. By the corollary of the preceding section, H is nilpotent. Hence G is locally nilpotent.

Next, assume G is not a torsion group. Let $a \in G$ have infinite order, and p be a prime. Then some ascendant subgroup lies between $\langle a \rangle$ and $\langle a^{p^2} \rangle$, therefore $\langle a^p \rangle$ is ascendant in G. Similarly, if q is any other prime, $\langle a^q \rangle$ is also ascendant, and therefore $\langle a \rangle = \langle a^q \rangle \langle a^q \rangle$ is ascendant [2, Theorem 2]. Hence $a \in R(G)$.

We show now that G is locally nilpotent, by proving that G is generated by its elements of infinite order, and hence $G = R(G)$.

If G is torsion-free, there is nothing to prove. Let, then, $1 \neq b \in G$ be a torsion element, and let a be an element of infinite order. Consider $H = \langle a, b \rangle$. $R(H)$ contains all elements of infinite order in H, therefore $H/R(H)$ is a finitely generated torsion group, which, if it is infinite, is locally nilpotent by the first part of the proof. Thus $H/R(H)$ is finite, $R(H)$ has finite index in H, and thus $R(H)$ itself is a finitely generated infinite nilpotent group. By [8, 7.1.11] $R(H)$ contains a torsion free subgroup, K, which is normal and of finite index in H.

Denote $Z = Z(K)$, and consider $\langle b, Z \rangle$. Let m be an integer, prime to the order of b, and divisible by four different primes at least. Since Z is free abelian of finite rank, $\langle b, Z \rangle / Z^m$ is a finite group with dense subnormal subgroups, which is nilpotent, by Theorem 1. As $\langle b \rangle Z^m / Z^m$ and Z/Z^m have relatively prime order, we obtain $[b, Z] \subseteq Z^m$. Taking an infinite sequence of such m's shows that b centralizes Z. Hence, if $k \neq z \in Z$, both z and bz have infinite order, and $b = bz \cdot z^{-1}$ belongs to the subgroup generated by the elements of infinite order in G. This proves our assertion, and completes the proof of Theorem 2.

3. Theorem 3. *Let G be a group with dense normal subgroups. If G contains an element of infinite order, then G is abelian.*

Proof. Let a be an element of infinite order. Let p and q be different primes. As in the proof of Theorem 2 $\langle a^q \rangle \lhd G$ and $\langle a^p \rangle \lhd G$, so $\langle a \rangle \lhd G$. Hence, for each $g \in G$, $g^{-1}ag = a$ or $g^{-1}ag = a^{-1}$.

Let b be another element of infinite order. If b does not commute with a, then $b^{-1}ab = a^{-1}$, and by symmetry, $a^{-1}ba = b^{-1}$. Hence, $b^{-2}ab^2 = b^{-1}a^{-1}b = a$, and a commutes with b^2, while $a^{-1}b^2a = b^{-2}$, a contradiction. So all elements of infinite order commute with a.

Let c be an element of finite odd order. If c does not commute with a, then $c^{-1}ac = a^{-1}$, $c^{-2}ac^2 = a$, so c^2 commutes with a. Since c is a power of c^2, c does commute with a.

Lastly, let c be an element of finite even order which does not commute with a. Again, $c^{-1}ac = a^{-1}$ and c^2 commutes with a. Hence c^2 is central in $\langle a, c \rangle$. Consider $\langle a, c \rangle / \langle c^2 \rangle = \langle \bar{a}, \bar{c} \rangle$, where $\bar{a} = a\langle c^2 \rangle$, $\bar{c} = c\langle c^2 \rangle$. This is also a group with dense normal subgroups. In particular, some normal subgroup, H, of $\langle \bar{a}, \bar{c} \rangle$ lies between $\langle \bar{c} \rangle$ and $\langle \bar{c}, \bar{a}^3 \rangle$. $\bar{c} \in H$ implies $\bar{a}^{-2}\bar{c} = \bar{a}^{-1}\bar{c}\bar{a} \in H$, so also $\bar{a}^{-2} \in H$, and hence $\bar{a}^2 \in \langle \bar{c}, \bar{a}^3 \rangle$, which is not true. So a commutes with all elements of of G, and is central in G.

In the same way we prove that each element of G of infinite order is central. Let $b \in G$ have finite order, then ab has infinite order, so a and ab are in the center of G, and so is $b = a^{-1}.ab$. So each element of G is in the center, i.e., G is abelian.

THEOREM 4. *Let G be an infinite group with dense normal subgroups. Then G is nilpotent, and of class 3 at most.*

Proof. By Theorem 3, one may assume that G is a torsion group. Theorem 2 implies that G is a locally finite group (we remark that, with the present stronger assumption, the proof of Theorem 2 can be modified, so as to avoid the use of Kostrikin's deep result. See proof of Theorem 7). Let H be any finite subgroup of G. One can add enough elements of H, so as to get another finite subgroup K, with $H \subset K$ and H not maximal in K. By assumption, there exists a subgroup L such that $H \subset L \subset K$ and $L \triangleleft G$. All the conjugates of H are contained in the finite group L, therefore H has only a finite number of conjugates in G.

Next, let H be an infinite subgroup of G. Let $a \in H$, then, by assumption, there exists a subgroup N_a satisfying $a \in N_a \subset H$ and $N_a \triangleleft G$. Since $H = \prod_{a \in H} N_a$, we obtain $H \triangleleft G$.

Thus, all the subgroups of G have only finitely many conjugates. By a result of B. H. Neumann [5], $|G:Z(G)|$ is finite. Therefore each subgroup containing $Z(G)$ is infinite, and, by the preceding paragraph, normal in G. Therefore in $G/Z(G)$ each subgroup is normal, and $G/Z(G)$ is a Dedekind group. In particular, $G/Z(G)$ is nilpotent of class $\leqq 2$, so G itself is nilpotent of class $\leqq 3$.

An example of an infinite group with dense normal subgroups which is not a Dedekind group is constructed as follows. Let p be an odd prime, and let A be a group of type $C(p^\infty)$. Let $B = \langle b \rangle$ be a group of order p. Adjoin to the direct product $A \times B$ an element c satisfying: $ca = ac$ for all $a \in A$, $bc = cbz$, where z is an element of order p in A, and $c^p = 1$. Then the group $G = \langle A, B, c \rangle$ has dense normal subgroups.

To see this, notice that $G' = \langle z \rangle$, hence any subgroup containing z is normal. Denote $D = \langle b, c \rangle$ then $z \in D$, D is the non-abelian group of order p^3 and exponent p, and $G = AD$. Let H be a subgroup of G not contained in D, and let $h \in H$, $h \notin D$. Write $h = ad$, $a \in A$, $d \in D$, then $h^p = a^p \neq 1$ (otherwise $a \in \langle z \rangle$ and $h \in D$). Therefore $\langle z \rangle \subseteq H$. On the other hand, if $H \subseteq D$ and $|H| \geqq p^2$, then again $z \in H$.

Now let $K \subset H$ be any two subgroups of G, with K non-maximal in H. Then $|H| \geqq p^2$, so our previous remarks show that $z \in H$ and hence $\langle K, z \rangle$ is a normal subgroup of G. If $z \notin K$, then this subgroup lies properly between H and K, while if $z \in K$ then any subgroup lying between H and K will do.

The next result shows, that this example is typical of the general case.

THEOREM 5. *Let G be an infinite p-group with dense normal subgroups.*

If G is not a Dedekind group, it contains a central subgroup C of type $C(p^\infty)$, such that G/C is a finite Dedekind group.

Proof. Since G is not a Dedekind group, there exists some cyclic subgroup, A say, which is not normal. Let $Z = Z(G)$ and $H = AZ$. As we have seen in the preceding proof, Z has finite index in G and all subgroups containing Z are normal. Therefore $H \lhd G$. Also $A \lhd H$ and $H/A \cong Z/Z \cap A$ is an infinite abelian p-group.

Let B/A be any subgroup of order p^2 of H/A. By assumption, there exists a subgroup K, $A \subset K \subset B$, such that $K \lhd G$. If D is any other subgroup such that $|D:A| = p$, then $A = K \cap D$. Since K is normal in G, while A is not, D is not normal in G. Therefore K is the only normal subgroup of G such that $A \subset K$ and $|K:A| = p$.

If K/A is the unique subgroup of order p in H/A, then $H/A \cong C(p^\infty)$. Assume D/A is another subgroup of order p in H/A. Then $H = DZ$, hence $D \lhd H$. Let $D \subset E \subset H$, $|E:D| = p$. Then some normal subgroup of G lies between and E, and by the previous paragraph, this is K. Since $D \neq K$, we have $E = DK$ and $E/D = KD/D$. Thus KD/D is the unique subgroup of order p in H/D, implying $H/D \cong C(p^\infty)$.

Thus, either $Z/A \cap Z$ or $Z/D \cap Z$ is isomorphic to $C(p^\infty)$, and $A \cap Z(D \cap Z)$ is finite. Hence Z satisfies the minimum condition. Being an abelian p-group, Z is a direct product of a finite group and a certain number of copies of $C(p^\infty)$. Since Z has a finite subgroup with quotient group isomorphic to $C(p^\infty)$, only one copy of $C(p^\infty)$ can appear in the decomposition of Z. Let C be this subgroup. Then $|Z:C|$, and hence also $|G:C|$, is finite. It was established in the proof of Theorem 4, that each infinite group of G is normal. Hence each subgroup containing C is normal, and G/C is a finite Dedekind group.

By appealing to Theorem 8, one sees that "p-group" in the formulation of Theorem 5 can be changed to "torsion group".

4. For the investigation of groups with dense quasi-normal subgroups, we need the following preliminary result (probably of independent interest).

THEOREM 6. *Let the group G be generated by quasi-normal subgroups, each of which is either infinite cyclic or finite and nilpotent. Then G is locally nilpotent.*

If, furthermore, all of the generating subgroups are finite, then G satisfies the normalizer condition.

Proof. The first assertion is equivalent to the following: a group generated by finitely many quasi-normal subgroups of the type described in the theorem is nilpotent. We shall prove the theorem in this form, beginning with the case of two factors.

Let, then, $G = HK$, where H and K are quasi-normal, and either infinite cyclic or finite nilpotent. First, if both H and K are finite, then G is finite, and H and K are subnormal nilpotent subgroups, so G is nilpotent.

Next, suppose H is infinite and K is finite. Then H has a finite index in G, therefore it contains a subgroup, H_1, also of finite index in G, such that $H_1 \lhd G$. Let p be a prime not dividing $|K|$. For each n, $H_1K/H_1^{p^n}$ is nilpotent (by the previous case), and finite, with $H_1/H_1^{p^n}$ as Sylow p-subgroup. Therefore K centralizes $H_1/H_1^{p^n}$. Since $\cap H_1^{p^n} = 1$, K centralizes H_1. Hence $H_1 \subseteq Z(G)$. G/H_1 is nilpotent, by the previous case, therefore G itself is nilpotent.

So now suppose H and K are both infinite cyclic. If $H \cap K \neq 1$, then $H \cap K \subseteq Z(G)$, and $G/H \cap K$ is finite, so G is nilpotent, as before. We may assume then $H \cap K = 1$, which means that each element of G is uniquely expressed as hk, with $h \in H$, $k \in K$.

Let $H = \langle h \rangle$, $K = \langle k \rangle$. Since $HK = KH$, $hk = k_1h_1$, with $k_1 \in K$, $h_1 \in H$. Let $K_1 = \langle k_1 \rangle$, then $hk \in K_1H = HK_1$. If $K_1 \neq K$, we get a contradiction to the uniqueness of the expression hk. Therefore $K_1 = K$ and $k_1 = k^{\pm 1}$. Similarly, $h_1 = h^{\pm 1}$.

Suppose $k_1 = k$. Then $hk = kh_1$, so $k^{-1}hk = h_1$, and $k^{-1}Hk = H$, $H \lhd G$. If $h_1 = h^{-1}$, then G/H^pK^2, where p is a prime $\neq 2$, is a dihedral group of order $2p$, which is not nilpotent, and thus contradicts the first portion of the proof. Therefore $h_1 = h$, and G is abelian.

Similarly, if $h_1 = h$, we find that $k_1 = k$ and G is abelian. There remains the case $hk = k^{-1}h^{-1}$. Since h^{-1} is also a generator of H, then, replacing h by h^{-1} in the above arguments, we see that we may also assume $h^{-1}k = k^{-1}(h^{-1})^{-1} = k^{-1}h$. But then $k^{-1}h^2k = k^{-1}h \cdot hk = h^{-1}k \cdot k^{-1}h^{-1} = h^{-2}$. Therefore $H^2 = \langle h^2 \rangle \lhd G$. For p as above, we find now that $H^2K/H^{2p}K^2$ is not nilpotent, which is again a contradiction. This concludes the proof for two factors (in the last part of the proof we have actually shown, that a product of two infinite cyclic quasi-normal subgroups with trivial intersection is abelian.)

Now let $G = H_1H_2 \cdots H_n$, with each H_i quasi-normal and either infinite cyclic or finite nilpotent subgroup. It follows from the proof for $n = 2$, that, for each pair (i,j), $|H_i : C_{H_i}(H_j)|$ is finite, (this holds also for $i=j$). Therefore, denoting $K_i = \bigcap_j C_{H_i}(H_j)$, $|H : K_i|$ is finite, and $|G : K_1K_2 \cdots K_n|$ is finite. Obviously, $K_i \subseteq Z(G)$, so $|G : Z(G)|$ is finite. Each of the subgroups $H_iZ(G)/Z(G)$ is quasi-normal and nilpotent in the finite group $G/Z(G)$, therefore each such subgroup is subnormal, so $G/Z(G)$, and with it G, is nilpotent.

We now prove the second assertion. Let G be generated by finite nilpotent quasi-normal subgroups. Let H be a subgroup of G. We may assume $H \neq G$. Therefore there exists a subgroup K, finite nilpotent and quasi-normal in G, such that $K \nsubseteq H$. H is of finite index in HK, therefore H contains a subgroup H_1 which is normal and of finite index in HK. By the first part of the theorem, G is locally nilpotent, hence KH/H_1 is nilpotent. Therefore $H/H_1 \lhd\lhd HK/H_1$, implying $H \lhd\lhd HK$ and, as $H \neq HK$, $H \neq N(H)$.

THEOREM 7. *Let G be an infinite group with dense quasi-normal subgroups. Then*

a. *If G is a torsion group, it satisfies the normalizer condition.*

b. *If G contains an element of infinite order, it is quasi-hamiltonian.*

Proof. Given any group H, denote by $Q(H)$ the product of all finite nilpotent quasi-normal subgroups of H. If we can prove that, for G a torsion group with dense quasi-normal subgroups $Q(G) \neq 1$, then the argument proving $G = R(G)$ in the proof of Theorem 2 yields that, under the same assumptions, $G = Q(G)$, hence a. follows from Theorem 6.

Let, then, G be a torsion group with dense quasi-normal subgroups, and assume $Q(G) = 1$. As in the proof of Theorem 2, this implies that each finite subgroup of G has prime order. Let $1 \neq g \in G$ and $h \in G$, $h \notin \langle g \rangle$. Let the primes p and q be the orders, respectively, of g and h. Let $H = \langle g, h \rangle$.

If $\langle g \rangle$ is not maximal in H, we can find a quasi-normal subgroup, K, such that $\langle g \rangle \subset K \subset H$. Then $H = \langle g, h \rangle = \langle h \rangle K$, therefore $|H:K| = q$. If $\langle g \rangle$ is not maximal in K, we find another quasi-normal subgroup, L, with $\langle g \rangle \subset L \subset K$. Then $H = \langle h \rangle L$, and $|H:L| = q$, implying $L = K$. Therefore $\langle g \rangle$ is maximal in K. We can replace H by K, to assume that $\langle g \rangle$ is maximal in H.

H being infinite, there must be some quasi-normal subgroup of G, M say, such that $\langle 1 \rangle \subset M \subset H$. If $M = \langle g \rangle$, then maximality of $\langle g \rangle$ in H implies $\langle g \rangle \lhd H$, hence $|H:\langle g \rangle|$ is finite, and H is finite. Therefore $M \neq \langle g \rangle$. Now the maximality of $\langle g \rangle$ implies that $H = \langle g \rangle M$, and $|H:M| = p$. This implies that M is infinite, and therefore there exists a quasi-normal subgroup, N, with $\langle 1 \rangle \subset N \subset M$. As above, we have $N \neq \langle g \rangle$, $H = \langle g \rangle N$, $|H:N| = p$ and therefore $M = N$. This is the sought-for contradiction.

We now prove b. Let G be a group with dense quasi-normal subgroups, and let g be an element of infinite order in G. As in Theorems 2 and 3, $\langle g \rangle$ is quasi-normal. Let h be any element of G of finite order. Then $\langle g \rangle$ has finite index in $\langle g, h \rangle = \langle g \rangle \langle h \rangle$. Hence, for some i, $\langle g^i \rangle \lhd \langle g, h \rangle$. Let n be an integer relatively prime to the order of h and divisible by 3 primes at least. Then Theorem 1 shows that $\langle g^i, h \rangle / \langle g^{ni} \rangle$ is nilpotent, hence $[g^i, h] \in \langle g^{ni} \rangle$. By taking a sequence tending to infinity of such n's we obtain $[g^i, h] = 1$. This implies $h = g^{-i} \cdot g^i h$, where both g^{-i} and $g^i h$ have infinite order. Therefore G is generated by its elements of infinite order.

Let H be any finite subgroup of G. We claim that H is contained in some finite quasi-normal subgroup. Assume this is not the case. Then, by the assumption of density, H is contained in a subgroup K which is a maximal finite subgroup of

G ($H = K$ or H is maximal in K). K contains a maximal subgroup, L say, which is quasi-normal in G. If M is any conjugate of L, then M is also quasi-normal, so maximality of K implies $K = KM$ and $M \subseteq K$. If K is generated by the conjugates of L, then $K \lhd G$, a contradiction to our assumption on H. Hence $L \lhd G$. We can pass to the group G/L, and hence assume that K has prime order.

Let g be an element of of infinite order. Then $\langle g \rangle$ is quasi-normal and maximal (of prime index) in $\langle g \rangle K$, hence $\langle g \rangle \lhd \langle g \rangle K$. The argument in the first paragraph of the proof, with g replacing g^i, shows that g centralizes K. As G is generated by the elements of infinite order, K is central in G, hence normal, a contradiction.

It follows, that the set of elements of finite order in G coincides with the product of all finite quasi-normal subgroups, hence it forms a characteristic subgroup F.

Let again g be an element of infinite order, and H a finite subgroup. Then $H = H\langle g \rangle \cap F$, so that $H \lhd H\langle g \rangle$ and $g \in N(H)$. As G is generated by all these g's, $H \lhd G$. Together with the fact that all infinite cyclic subgroups of G are quasi-normal, this shows that all cyclic subgroups, hence all subgroups, are quasi-normal.

5. Let G be an infinite torsion group with dense subnormal subgroups. By Theorem 2, G is locally nilpotent, and hence a direct product of its Sylow subgroups. We now discuss this decomposition. Everything in this section remains true if we substitute either "normal", "quasi-normal" or "ascendant" for "subnormal" throughout.

DEFINITION. A group G is said to have property (P) if, whenever H and K are subgroups of G, with H maximal in K, then either H or K is subnormal in G.

THEOREM 8. *Let G be an infinite torsion group with dense subnormal subgroups. Then at least one of the following holds*:

a. *G is a p-group.*

b. *Each subgroup of G is subnormal.*

c. *$G = H \times C_r$, where H is a p-group having property (P), C_r is a group of order r, and r is a prime different from p.*

Conversely, if $G = H \times C_r$, where H is a p-group having property (P) and with dense subnormal subgroups, C_r is a group of order r, r is a prime, and $r \neq p$, then G has dense subnormal subgroups.

Proof. Let the decomposition of G into a direct product of its Sylow subgroups be

$$G = G_{p_1} \times G_{p_2} \times \cdots$$

the p_i's are different primes, and G_{p_i} is the p_i-Sylow subgroup of G. We assume that $G_{p_i} \neq 1$ for each i. Suppose neither a. nor b. holds. Then there exists some subgroup H of G which is not subnormal. H is also a direct product of its Sylow subgroups, and at least one of these is not subnormal in G. Suppose H_{p_1}, the p_1-Sylow subgroup of H, is not subnormal. Let K be a subgroup of order pq of G, where p and q are primes different from p_1 (but $p = q$ is possible). Then some subnormal subgroup of G, L say, lies between H_{p_1} and $H_{p_1} \times K$. H_{p_1} is the p_1-Sylow subgroup of L, therefore H_{p_1} is subnormal in G. This contradiction shows that no such K exists, which is possible only if $G = G_{p_1} \times C_r$, $r \neq p$. Let M and N be subgroups of H_{p_1}, with M maximal in N. Then some subnormal subgroup of G lies between M and $N \times C_r$, and this subgroup can be only N or $M \times C_r$. Therefore either M or N is subnormal in G, showing that H has property (P), and that G satisfies c.

The converse part of Theorem 7 is easily verified. It is natural to ask whether groups with property (P) have dense subnormal subgroups. Let us call a group a *T-group*, if it is infinite and each proper subgroup of it has prime order. It is not known if such groups exist (the question of their existence is attributed in [6] to Tarski). A T-group, if one exists, would be a group with property (P) and not having dense subnormal subgroups (or any subnormal subgroups). Let us show that this is essentially the only obstacle.

PROPOSITION. *Let G be a group with property (P). If no section of G is a T-group, then G has dense subnormal subgroups.*

Proof. Let H and K be subgroups of G such that $H \subset K$, but H is not maximal in K. Suppose no subnormal subgroup of G lies between H and K. Let $a \in K - H$. Then H is contained in some maximal subgroup, L say, of $H_1 = \langle H, a \rangle$. By property (P), either L or H_1 is subnormal. Since H is not maximal in K, we may choose a such that $H_1 \neq K$. Therefore $L = H$ and $H \lhd\lhd G$, otherwise either L or H_1 would be a subnormal subgroup lying between H and K. Now choose $b \in K - H_1$ and let M be a maximal subgroup of $H_2 = \langle H, a, b \rangle$ containing H_1. Similar reasoning yields $H_2 = K$ and $K \lhd\lhd G$. Also $M = H_1$, or we could choose $b \in M$ and obtain $H_2 \neq K$.

If some subnormal subgroup of K contains H, then this subgroup is also subnormal in G, therefore it is either H or K. Hence H is a maximal subnormal subgroup of K, forcing $H \lhd K$, and also showing that K/H is simple. If N/H is any proper subgroup of K/H, we may choose $a \in N$ in the preceding paragraph, so that $H_1 \subset N$. As H_1 is maximal in K, $H_1 = N$, and N/H is maximal in K/H. Since each subgroup of the torsion group K/H is maximal, each such subgroup has prime order. If K/H were finite, it would have order p^2 or pq (p, q primes) and thus would not be simple. Therefore K/H is a T-group, a contradiction. This concludes the proof.

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